Solution of the Lane-Friedman Integral Equation

Jiri Stern*
Aeronautical Research and Test Institute, Prague, Czechoslovakia

Solution of the Lane-Friedman[1] integral equation is completed, which gives partly new numerical techniques for computing force and moment coefficients for the following structures: 1) aerofoils oscillating in a given manner in a staggered cascade, 2) a single wing oscillating harmonically between two parallel walls, and 3) a single wing oscillating in a freestream. A common method for computing the critical flutter speed of these structures is then given. One of the author's contribution to this problem lies in the derivation of a method for computing a set of matrix elements, which reduces the solution of a particular case of this integral equation (isolated wing) to the solution of a set of linear algebraic equations. However, the author's most significant contribution to this method lies in the fact that the sum of this square matrix and another matrix (derived in Ref. 1), also gives a similar method for the solution of the general integral equation. Thus, the first square matrix, whose derivation is not incorporated in Ref. 1, represents the heart of the general method. Since the solution of the first integral equation represents a new numerical method, even in unsteady single-wing theory, its accuracy has been checked by some results tabulated by different methods of a number of authors (Reissner-Haskind, Dietze, etc.). For a range of k between 0.02 and 1.0 excellent agreement has been obtained in all cases. The method is based on the assumption of two-dimensional compressible subsonic flow.

Nomenclature

A_j = ($j=0,1,\ldots$) coefficient in pressure-jump expansion (nondimensional)
A_{jF}, A_{jM} = ($j=0,1,\ldots$) values of A_j associated with constant and linearly varying upwash, respectively (nondimensional)
ab = elastic axis location measured from midchord (Fig. 2), m
b = semichord (Fig. 2), m
C'_h = ($m'\omega_h^2$) flexural spring constant per unit length (span), N/m^2
C'_t = ($I'_p\omega_t^2$) torsional spring constant per unit length (span), N/m − rad
c = chord m
d = ($\bar{M}r_g^2$) dimensionless mass moment of inertia about elastic axis
F' = resultant aerodynamic force per unit length (span) related to elastic axis (Fig. 2), N/m
h = bending deflection of elastic axis (Fig. 2), m
h_0 = maximum amplitude of h, m
Im = imaginary part of a function
I'_p = ($m'r_g b^2$) mass moment of inertia per unit length (span) about elastic axis, kgm
i = imaginary unit
J_j = Bessel function of the order j
k = ($\omega b/V$) reduced frequency (nondimensional)
L'_F = oscillatory aerodynamic force per unit length (span) due to translation, N/m
L'_M = oscillatory aerodynamic force per unit length (span) due to pitch, N/m
M = freestream Mach number (nondimensional)
$\bar{M}$ = (m'/m'_a) dimensionless mass ratio
M'_a = resultant moment of aerodynamic forces about elastic axis, per unit length, N
M'_F = oscillatory midchord moment per unit length (span) due to translation, N
M'_M = oscillatory midchord moment per unit length (span) due to pitch, N
m' = mass of the blade (wing) per unit length (span), kg/m
m'_a = mass of surrounding air cylinder per unit length, kg/m
Re = real part of a function
rb = location of center of gravity of the blade (wing) measured from elastic axis (Fig. 2), m
r_gb = (I'_p/m') radius of gyration referred to elastic axis, m
S' = ($m'r\ b$) static mass moment per unit length (span) about elastic axis, kg
s_1 = gap distance (Fig. 1); height of tunnel, m
s_2 = stagger distance (Fig. 1), m
t = time, sec
$u_0(x)$ = upwash contribution corresponding to the single-wing term [Eq. (1)], m/sec
V = freestream velocity, m/sec
$v_0(x)$ = upwash contribution corresponding to the cascade term [Eq. (1)], m/sec
$w_0(x)$ = total upwash corresponding to the general integral equation [Eq. (1)], m/sec
x = streamwise coordinate on the blade (wing), nondimensional
β = ($[1-M^2]^{1/2}$) auxiliary parameter (nondimensional)
$\bar{\beta}$ = interblade phase-lag angle for $M\neq 0$, rad
δ = stagger angle (Fig. 1), rad
ϵ = ($\bar{M}r$) dimensionless mass static moment about elastic axis
θ = angular deflection about elastic axis (Fig. 2)
θ_0 = maximum amplitude of θ, rad
κ = (kM/β^2) auxiliary parameter, nondimensional
ρ_0 = freestream density, kg/m^3
τ = variable in Eq. (1)
ω = frequency of vibration, rad/sec
ω_h = fundamental bending frequency of the blade (wing), rad/sec
ω_t = fundamental torsional frequency of the blade (wing), rad/sec

Superscripts

$\ddot{h}$ = second time derivative of h
$\ddot{\theta}$ = second time derivative of θ

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*Research engineer, Mathematical Department.

Introduction

THE investigation of the flutter phenomenon of an infinite staggered cascade of thin aerofoils oscillating harmonically with constant phase-lag angle between adjacent blades in two-dimensional compressible subsonic flow can be reduced to two basic problems—mechanical and aerodynamic.

The solution of the aerodynamic problem was first given by Lane and Friedman.[1] They have shown that the calculation of these oscillatory aerodynamic forces and moments can be reduced to the solution of an integral equation [Eq. (1a)] which relates the known upwash distribution over the blade to the unknown pressure-jump distribution. Moreover, Lane and Friedman have also shown that it is very useful to write the resultant upwash as a sum of two quantities [Eqs. (1b) and (1c)] of which the first corresponds to the oscillation of the reference blade (single-wing term) and the second to the oscillation of other blades (cascade term). For $\bar{\beta}=\pi$ and $\delta=0$, the cascade term, together with the corresponding single-wing term, enables us to treat the case of a wing oscillating harmonically between two parallel walls. Finally, if we neglect the cascade term, we obtain a simpler integral equation [Eq. (1b)] whose solution gives analogous force and moment coefficients for a single wing oscillating in freestream.

The general integral equation will be solved by the collocation method, which reduces its solution to that of a set of linear algebraic equations. It is evident that the resultant square matrix of this set of linear equations is equal to the sum of two square matrices, corresponding to the single-wing term and cascade term, respectively. Thus the solution of the general integral equation is dependent on the elements of the aforementioned two square matrices.

Having reviewed the most important results of Ref. 1, we are now in a position to list its disadvantages and to formulate the purpose of this paper. Reference 1, based on Fourier transform techniques, represents an excellent derivation of the general integral equation. However, in the case of a compressible subsonic flow, it gives no complete solution. Three disadvantages of Ref. 1 can be listed: 1) It deals only with the evaluation of the elements of the second square matrix, corresponding to the cascade term, but it gives no analogous method for evaluating the elements of the first square matrix, corresponding to the single-wing term; 2) It gives no tables of unsteady aerodynamic coefficients for the case of a staggered cascade; 3) It does not deal with calculating the critical flutter speed of the aforementioned three structures.

In this paper the results of Ref. 1 are first extended to give numerical techniques for computing force and moment coefficients for the previously mentioned structures. Then a common method for calculating the critical flutter speed of these structures is given.

The author's most significant contribution lies in the derivation of a method for computing the elements of the first square matrix, corresponding to the single-wing term. This matrix, together with the second matrix, can be used either as a contribution to the above-mentioned resultant square matrix, corresponding to the solution of the general integral equation, or as another set of linear algebraic equations corresponding to the solution of a single wing oscillating in free stream. The last integral equation differs from those derived by other authors, since its kernel is given in closed form. Therefore, it was necessary to check the accuracy of the author's solution of this integral equation by the results of other investigators.

In Appendix A, the method is compared with some results tabulated by Blanch and Fettis (corrected Reissner-Haskind method[2]), Dietze,[2] Turner and Rabinowitz,[2] as well as by Frazer[3] (improved Possio's method), and for a range of k between 0.02 and 0.7 excellent agreement has been obtained in all cases. For $k=1$, our results agree very well also with those given by Frazer.[3] However, for $k>1$ the results differ from those of Frazer, as well as Blanch and Fettis, but the author believes that the method holds also for large values of k. The latter discrepancies are due to the fact that the solutions of the corresponding integral equations have been performed by the seven-point collocation method, which is unsatisfactory for large values of k.

In Appendix B, the results are checked with those of Woolston and Runyan.[4] In this case (oscillating wing between two parallel walls) also very good agreement has been obtained. In Appendix C, the case of a staggered cascade is treated and the Lane's[5] minimization procedure is carried out by using aerodynamic coefficients tabulated in the subresonant zone.

Note that, more recently, the two-dimensional aerodynamic problem was studied by a number of investigators,[6-8] but their methods are based on the well-known methods used in unsteady single-wing theory. In fact, the results of Kurzin[6] and Smith[7] are based on the improved Possio's method (use of Hankel functions) and the results of Gorelow and Dominas[8] are based on the Reissner-Haskind method (use of Mathieu functions). Thus the completed Lane-Friedman method is superior to them, since it represents a new method not only for an oscillating cascade, but also for a single wing, oscillating in freestream. It is worthy of mention that Gorelow[9] gave a three-dimensional method for calculating force and moment coefficients for a cascade of oscillating membranes, but he derived no method for calculating the critical flutter speed of this structure. Moreover, as far as the writer knows, this problem is still under study.

The work of Whitehead[10] is based also on the solution of the Lane-Friedman integral equation. This fact will be discussed later.

Governing Equations

Lane and Friedman have shown that the aerodynamic problem can be reduced to the solution of an integral equation of the form [see Eq. (55) of Ref. 1]

$$w_0(x)/V=[u_0(x)/V]+[v_0(x)/V]=\text{single-wing contribution}+\text{cascade contribution} \tag{1a}$$

where

$$\frac{u_0(x)}{V}=\lim_{k_2\to 0^-}\frac{\beta}{2}e^{iMx\kappa}\times\int_{\tau=-\infty}^{\infty}\frac{[e^{i\tau x}\Gamma_0(\tau)(\kappa^2-\tau^2)^{1/2}]}{\tau+k/\beta^2}\,0/\tau \tag{1b}$$

where $\kappa=kM/\beta^2$, and

$$\frac{v_0(x)}{V}=\lim_{k_2\to o^-}\frac{\beta}{2}e^{ikM^2x/\beta^2}\times\int_{\tau=-\infty}^{\infty}\frac{e^{i\tau x}\Gamma_0(\tau)}{\tau+k/\beta^2}\left[\frac{k^2M^2}{\beta^4}-\tau^2\right]^{1/2}$$
$$\times\left[\frac{i\sin[\frac{s_1\beta}{b}(\frac{k^2M^2}{\beta^4}-\tau^2)^{1/2}]}{\cos[\frac{s_1\beta}{b}(\frac{k^2M^2}{\beta^4}-\tau^2)^{1/2}]-\cos(\bar{\beta}-\frac{s_2\tau}{b})}-1\right]\mathrm{d}\tau \tag{1c}$$

where for $(kM/\beta^2)^2<\tau^2$ we must take $-i(k^2M^2/\beta^4-\tau^2)^{1/2}$ in place of $i(k^2M^2/\beta^4-\tau^2)^{1/2}$.

In Eqs. (1b) and (1c) the quantity k is a complex number

$$k=k_1+ik_2\quad 0<|-k_2|\ll 1$$

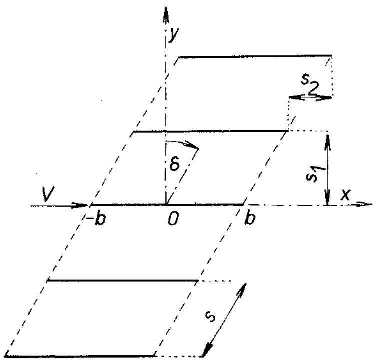

Fig. 1 Cascade geometry.

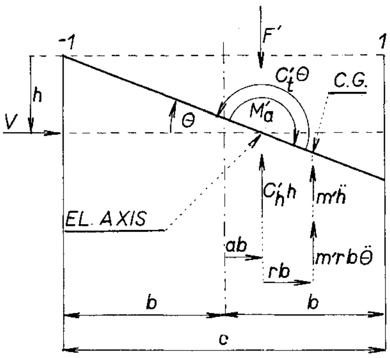

Fig. 2 Force and moments acting on the reference blade.

and [see Eq. (42) of Ref. 1]

$$\Gamma_0(\tau)=\tfrac{1}{2}\Big\{\Big[A_0+\frac{A_1}{2}\Big]J_0\Big[\tau+\frac{kM^2}{\beta^2}\Big]$$

$$+\tfrac{1}{2}\sum_{j=2}^{\infty}(i)^j(A_{j+1}-A_{j-1})J_j\Big[\tau+\frac{kM^2}{\beta^2}\Big]\Big\} \tag{2}$$

The symbols appearing in Eqs. (1b, 1c, and 2) are defined in the Nomenclature (see also Fig. 1).

Lane and Friedman have shown that Eq. (1a) can be solved only in the regions labeled A (subresonant zone), B, C, D,... (superresonant zones) in Fig. 2 of Ref. 1, the two sets of lines being the zone boundaries at which solution of this integral equation does not exist. In Ref. 1 it is shown that blade-row flutter would probably occur with values of β between $\pi/4$ and $7\pi/4$ and with critical conditions falling in the subresonant zone. Therefore, the coefficients of unsteady aerodynamic forces and moments have to be calculated first of all in zone A. From Eq. (1a) it can be seen that the upwash $w_0(x)$ ($|x|<1$) is equal to the sum of two functions, the first being called "single-wing term" and the second the "cascade term." From Eqs. (1b) and (1c) it can be seen that the general integral equation can be solved numerically for each of the aforementioned structures.

Equation (1a) [and also Eq. (1b)] has to be solved under two conditions: a) $w_0(x)/V=1$ [$u_0(x)/V=1$] (translation of midchord) and b) $w_0(x)/V=x$ [$u_0(x)/V=x$] (pitch). The corresponding integral equation will be solved by the collocation method. Once the coefficients have been computed, the oscillatory force and midchord moment can be written as follows

$$L'_F/\pi\rho_0V^2b=-[A_{0F}+(A_{1F/2)}] \tag{3a}$$

$$M'_F/\pi\rho_0V^2b^2=-\tfrac{1}{2}[A_{0F}+(A_{2F}/2)] \tag{3b}$$

Similarly

$$L'_M/\pi\rho_0V^2b=-[A_{0M}+(A_{1M}/2)] \tag{4a}$$

$$M'_M/\pi\rho_0V^2b^2=-\tfrac{1}{2}[A_{0M}+(A_{2M}/2)] \tag{4b}$$

Equations (3) and (4) enable us to calculate analogous force and moment amplitudes related to the elastic axis of the blade (wing). Using the notation of Fig. 2, we obtain

$$\frac{F'}{\pi\rho_0V^2b}=-\frac{h}{b}\Big\{ik\Big[A_{0F}+\frac{A_{1F}}{2}\Big]\Big\}$$

$$-\theta\Big\{(1-ika)\Big[A_{0F}+\frac{A_{1F}}{2}\Big]+ik\Big[A_{0M}+\frac{A_{1M}}{2}\Big]\Big\} \tag{5}$$

$$\frac{M'_a}{\pi\rho_0V^2b^2}=-\frac{aF'}{\pi\rho_0V^2b}+\frac{h}{b}\Big\{\frac{ik}{2}\Big[A_{0F}+\frac{A_{2F}}{2}\Big]\Big\}$$

$$+\theta\Big\{\Big(\frac{1}{2}-\frac{ika}{2}\Big)\Big[A_{0F}+\frac{A_{2F}}{2}\Big]$$

$$+\frac{ik}{2}\Big[A_{0M}+\frac{A_{2M}}{2}\Big]\Big\} \tag{6}$$

To avoid the definition of a numer of new symbols we denote the oscillatory aerodynamic forces and moments corresponding to these different structures by common symbols F' and M'_a. Their meanings will be evident from the context. (See Fig. 2)

Single-Wing term

For practical computational purposes we divide the interval $<-\infty,\infty>$ into three subintervals $<-\infty,-N>$, $<-N,N>$ and $<N,\infty>$. By using the substitution $\tau=z-\kappa M$, and the abbreviation

$$g(z)=-(i\beta/2)e^{ixz}[(z-\kappa M)^2-\kappa^2]^{1/2}(|x|<1$$

$$[z-\kappa M>\kappa) \tag{7}$$

Eq. (1b) can be written as follows

$$u_0(x)/V=u_N(x)+u_r(x) \tag{8}$$

where

$$u_N(x)=\int_{-N}^{N}\frac{g(z)\Gamma_0(z-\kappa M)-g(-k)\Gamma_0(-k-\kappa M)}{z+k}dz$$

$$+g(-k)\Gamma_0(-k-\kappa M)\lim_{k_2\to0^-}\Big[\int_{-N}^{N}\frac{(z+k_1)dz}{(z+k_1)^2+k_2^2}$$

$$-ik_2\int_{-N}^{N}\frac{dz}{(z+k_1)^2+k_2^2}\Big]$$

$$=\int_{-N}^{N}\frac{g(z)\Gamma_0(z-\kappa M)-g(-k)\Gamma_0(-k-\kappa M)}{z+k}dz$$

$$+g(-k)\Gamma_0(-k-\kappa M)\ \Big(\ln\frac{N+k}{N-k}+i\pi\Big) \tag{9}$$

and

$$u_r(x)=\int_{N}^{\infty}\frac{g(z)\Gamma_0(z-\kappa M)}{z+k}dz+\int_{-\infty}^{-N}\frac{g(z)\Gamma_0(z-\kappa M)}{z+k}dz$$

$$=\int_{N}^{\infty}\frac{g(z)\Gamma_0(z-\kappa M)}{z+k}dz$$

$$+\int_{N}^{\infty}\frac{g(-z)\Gamma_0(-z-\kappa M)}{-z+k}dz \tag{10}$$

Equations (9) and (10) hold for any number of collocation points. Especially for a three-point collocation method, the following equations are valid [see Eq. (2)]

$$\Gamma_0(z-\kappa M) = \tfrac{1}{2}\{A_0[J_0(z)+iJ_1(z)] + (A_1/2)[J_0(z)+J_2(z)] + (iA_2/2)[J_1(z)+J_3(z)]\} \tag{11}$$

$$\Gamma_0(-k-\kappa M) = \tfrac{1}{2}\{A_0[J_0(-k)+iJ_1(-k)] + (A_1/2)[J_0(-k)+J_2(-k)] + (iA_2/2)[J_1(-k)+J_3(-k)]\} \tag{12}$$

In Eqs. (9-12) we assume that $k_1 \equiv k$. If we substitute for $\Gamma_0(z-\kappa M)$ from Eq. (11) and for $\Gamma_0(-z-\kappa M)$ from Eq. (12), Eq. (9) becomes

$$u_N(x) = A_0\left\{g(-k)\,\frac{J_0(-k)+iJ_1(-k)}{2}\left(\ln\frac{N+k}{N-k}+i\pi\right) + \tfrac{1}{2}\int_{-N}^{N}\frac{g(z)[J_0(z)+iJ_1(z)] - g(-k)[J_0(-k)+iJ_1(-k)]}{z+k}\,dz\right\}$$

$$+A_1\left\{g(-k)\,\frac{J_0(-k)+J_2(-k)}{4}\left(\ln\frac{N+k}{N-k}+i\pi\right) + \tfrac{1}{4}\int_{-N}^{N}\frac{g(z)[J_0(z)+J_2(z)] - g(-k)[J_0(-k)+J_2(-k)]}{z+k}\,dz\right\}$$

$$+iA_2\left\{g(-k)\,\frac{J_1(-k)+J_3(-k)}{4}\left(\ln\frac{N+k}{N-k}+i\pi\right) + \tfrac{1}{4}\int_{-N}^{N}\frac{g(z)[J_1(z)+J_3(z)] - g(-k)[J_1(-k)+J_3(-k)]}{z+k}\,dz\right\} \tag{13}$$

The integrands in Eq. (13) are now continuous functions also at $z=-k$ and, therefore, it is not necessary to indicate the Cauchy principal values of the corresponding integrals. Therefore, the integrals can be evaluated using, for example, the Gaussian nine-point formula. It can be seen that if results with higher accuracy are required, the interval $<-N,N>$ must be subdivided into many subintervals, and each integral has to be evaluated by the Gaussian formula. To evaluate $u_r(x)$ we expand

$$H(z) = \frac{[(z-\kappa M)^2-\kappa^2]^{1/2}}{z+k} \quad \text{and} \quad I(z) = \frac{[(z+\kappa M)^2-\kappa^2]^{1/2}}{k-z} \tag{14}$$

into two Taylor series at $z\to\infty$. These series are equivalent to those of the functions $H(1/y)$ and $I(1/y)$ at $y\to 0$. The first two derivatives of $H(1/y)$ at $y=0$ are

$$H'\left(\frac{1}{y}\right) = -\frac{k}{\beta^2}; \quad H''\left(\frac{1}{y}\right) = k^2\left(2+\frac{2M^2}{\beta^2}-\frac{M^2}{\beta^4}\right) = 2\zeta \tag{15}$$

Similarly

$$I'\left(\frac{1}{y}\right) = -\frac{k}{\beta^2}; \quad I''\left(\frac{1}{y}\right) = -k^2\left(2+\frac{2M^2}{\beta^2}-\frac{M^2}{\beta^4}\right) = -2\zeta \tag{16}$$

If we neglect the higher order terms in these series, Eq. (10) now becomes

$$\frac{2u_r(x)}{i\beta} \doteq -\int_N^\infty\left\{\left[1-\frac{k}{\beta^2 z}+\frac{\zeta}{z^2}-\ldots\right]e^{ixz}\Gamma_0(z-\kappa M) - \left[1+\frac{k}{\beta^2 z}+\frac{\zeta}{z^2}+\ldots\right]e^{-ixz}\Gamma_0(-z-\kappa M)\right\}dz \tag{17}$$

From Eq. (11) it is evident that the real and imaginary parts of $\Gamma_0(z-\kappa M)$ consist of Bessel functions of even and odd order, respectively. Therefore

$$\Gamma_0(-z-\kappa M) = \overline{\Gamma_0(z-\kappa M)}$$

where $\overline{}$ represents the complex conjugate. This identity holds also for the general case. Since

$$e^{-ixz}\,\overline{\Gamma_0(z-\kappa M)} = \overline{e^{ixz}\Gamma_0(z-\kappa M)} = \mathrm{Re}[e^{ixz}\Gamma_0(z-\kappa M)] - i\,\mathrm{Im}[e^{ixz}\Gamma_0(z-\kappa M)]$$

Equation (17) can be written in the form

$$u_r(x) \doteq \beta\int_N^\infty \mathrm{Im}[e^{ixz}\Gamma_0(z-\kappa M)]\,dz + \frac{ik}{\beta}\int_N^\infty\frac{\mathrm{Re}[e^{ixz}\Gamma_0(z-\kappa M)]}{z}\,dz$$

$$+\frac{\beta k^2}{2}\left(2+\frac{2M^2}{\beta^2}-\frac{M^2}{\beta^4}\right)\int_N^\infty\frac{\mathrm{Im}[e^{ixz}\Gamma_0(z-\kappa M)]}{z^2}\,dz \tag{18}$$

Dividing the function $e^{ixz}\Gamma_0(z-\kappa M)$ into real and imaginary parts and substituting them into Eq. (18), we have

$$u_r(x) \doteq A_0\left[\beta\sum_1 - \frac{ik}{2\beta}\sum_2 + \frac{\beta\zeta}{2}\sum_3\right]$$

$$+A_1\left[\beta\sum_4 + \frac{ik}{4\beta}\sum_5 + \frac{\beta\zeta}{4}\sum_6\right] + A_2\left[\beta\sum_7 - \frac{ik}{4\beta}\sum_8 + \frac{\beta\zeta}{4}\sum_9\right] \tag{19}$$

where

$$\sum_1 = \int_0^\infty \frac{J_0(z)\sin xz + J_1(z)\cos xz}{2} dz - \int_0^N \frac{J_0(z)\sin xz + J_1(z)\cos xz}{2} dz \tag{20}$$

$$\sum_2 = \int_0^\infty \frac{J_1(z)\sin xz + J_2(z)\cos xz}{z} dz + 2\int_0^\infty \frac{J_3(z)\cos xz}{z^2} dz$$

$$- \int_0^N \frac{J_1(z)\sin xz + J_2(z)\cos xz}{z} dz - 2\int_0^N \frac{J_3(z)\cos xz}{z^2} dz \tag{21}$$

$$\sum_3 = -\int_0^\infty \frac{J_3(z)\cos xz + J_2(z)\sin xz}{z^2} dz + \int_0^N \frac{J_3(z)\cos xz + J_2(z)\sin xz}{z^2} dz \tag{22}$$

$$\sum_4 = ½\int_0^\infty \frac{J_1(z)\sin xz}{z} dz - ½\int_0^N \frac{J_1(z)\sin xz}{z} dz \tag{23}$$

$$\sum_5 = -2\int_0^\infty \frac{J_3(z)\cos xz}{z^2} dz + 2\int_0^N \frac{J_3(z)\cos xz}{z^2} dz \tag{24}$$

$$\sum_6 = \sum_9 \doteq 0 \tag{25}$$

$$\sum_7 = \int_0^\infty \frac{J_2(z)\cos xz}{z} dz - \int_0^N \frac{J_2(z)\cos xz}{z} dz \tag{26}$$

$$\sum_8 = 4\int_0^\infty \frac{J_2(z)\sin xz}{z^2} dz - 4\int_0^N \frac{J_2(z)\sin xz}{z^2} dz \tag{27}$$

Equations (20-27) were reduced to conventional forms by using the well-known reduction formula. The results were simplified by neglecting all integrals, whose integrands are divided by z^3. Therefore, the method holds only for sufficiently large values of N. The infinite integrals appearing in Eqs. (20-27) are given by the following set of general equations $(0 < |x| < 1)$

$$\int_0^\infty J_u(z)\sin xz\, dz = \frac{\sin(u \sin^{-1}x)}{(1-x^2)^{½}} \qquad \mathrm{Re}\, u > -2 \tag{28}$$

$$\int_0^\infty J_u(z)\cos xz\, dz = \frac{\cos(u \sin^{-1}x)}{(1-x^2)^{½}} \qquad \mathrm{Re}\, u > -1 \tag{29}$$

$$\int_0^\infty \frac{J_u(z)\sin xz}{z} dz = \frac{1}{u}\sin(u \sin^{-1}x) \qquad \mathrm{Re}\, u > -1 \tag{30}$$

$$\int_0^\infty \frac{J_u(z)\cos xz}{z} dz = \frac{1}{u}\cos(u \sin^{-1}x) \qquad \mathrm{Re}\, u > 0 \tag{31}$$

$$\int_0^\infty \frac{J_u(z)\sin xz}{z^2} dz = \frac{(1-x^2)^{½}\sin(u \sin^{-1}x)}{u^2-1} - \frac{x\cos(u \sin^{-1}x)}{u(u^2-1)} \qquad \mathrm{Re}\, u > 0 \tag{32}$$

$$\int_0^\infty \frac{J_u(z)\cos xz}{z^2} dz = \frac{\cos[(u-1)\sin^{-1}x]}{2u(u-1)} + \frac{\cos[(u+1)\sin^{-1}x]}{2u(u+1)} \qquad \mathrm{Re}\, u > 1 \tag{33}$$

The other integrals can be evaluated again according to the Gaussian nine-point formula.

Cascade Term

The method for investigating the contribution of the cascade term to the total upwash is very simple. From Eq. (1c) it is evident that the bracketed term appearing in this equation converges exponentially to zero. Therefore, the limits of integration $< -\infty, \infty >$ can be replaced by finite numbers $< -N_I, N_I >$. Following the method outlined in the previous section, we obtain

$$\frac{v_0(x)}{V} = \frac{\beta}{2}\exp(ikM^2x/\beta^2)\left[G\left(\frac{-k}{\beta^2}\right)\left(i\pi + \ln\frac{-N_I + (k/\beta^2)}{-N_I - (k/\beta^2)}\right) + \int_{-N_I}^{N_I} \frac{G(\tau) - G(-k/\beta^2)}{\tau + k/\beta^2} d\tau\right] \tag{34}$$

where [see Eq. (1c)]

$$G(\tau) = \exp[i\tau x\Gamma_0(\tau)][(k^2M^2/\beta^4) - \tau^2]^{½} \times \left[\frac{i\sin[\frac{s_I\beta}{b}(\frac{k^2M^2}{\beta^4} - \tau^2)^{½}]}{\cos[\frac{s_I\beta}{b}(\frac{k^2M^2}{\beta^4} - \tau^2)^{½}] - \cos(\bar\beta - \frac{\tau s_2}{b})} - 1\right] \tag{35}$$

The evaluation of Eqs. (34) and (35) can be directly programed in Fortran language for computers, which enables us to avoid the difficulties, based on the evaluation of equations, derived in Appendix C of Ref. 1.

The method given in this section holds only in the subresonant zone, labeled A in Fig. 2 of Ref. 1, but it can be extended also to superresonant zones B,C,D,... (see pp. 25-28 of Ref. 1).

Critical Flutter Speed

If we substitute for $F'/(\pi\rho_0 bV^2)$ from Eq. (5) into Eq. (6) and use the relations

$$m'_a = \pi\rho_0 b^2 \quad k=\omega b/v \quad 2b=c \tag{36}$$

and

$$h = h_0\exp^{(i\omega t)} \quad \theta=\theta_0\exp[i(\omega t-\varphi)] \tag{37}$$

Equations (5) and (6) can be rewritten as follows

$$F' = -m'_a\omega^2\exp(i\omega t)\left[C_1 h_0 + \frac{c}{2}C_2\theta_0\exp(-i\varphi)\right] \tag{38}$$

$$M'_a = -m'_a\omega^2\frac{c}{2}\exp(i\omega t)\left[C_3 h_0 + \frac{c}{2}C_4\theta_0\exp(-i\varphi)\right] \tag{39}$$

where

$$C_1 = \frac{i}{k}\left[A_{0F} + \frac{A_{1F}}{2}\right] \tag{40}$$

$$C_2 = \frac{1-ika}{k^2}\left[A_{0F}+\frac{A_{1F}}{2}\right] + \frac{i}{k}\left[A_{0M}+\frac{A_{1M}}{2}\right] \tag{41}$$

$$C_3 = -\frac{i}{k}\left[(\tfrac{1}{2}+a)A_{0F} + \frac{A_{2F}+2aA_{1F}}{4}\right] \tag{42}$$

$$C_4 = -\frac{i}{k}\left[(\tfrac{1}{2}+a)A_{0M}+\frac{A_{2M}+2aA_{1M}}{4}\right] - \frac{1-ika}{k^2}\left[(\tfrac{1}{2}+a)A_{0F}+\frac{A_{2F}+2aA_{1F}}{4}\right] \tag{43}$$

If the notations of Fig. 2 are used and the time derivatives are denoted by dots, the equations of motion can be written as follows

$$m'\ddot{h} + C'_h h + S'\ddot{\theta} - F' = 0 \tag{44a}$$

$$S'\ddot{h} + C'_t\theta + I'_p\ddot{\theta} - M'_a = 0 \tag{44b}$$

Equations (44) can be solved by the well-known Theodorsen's method.[11]

Conclusions

The solution of the Lane-Friedman upwash integral equation is completed, which gives partly new numerical techniques for computing force and moment coefficients for the following structures: 1) airfoils oscillating in a given manner in a staggered cascade, 2) a single wing oscillating harmonically between two parallel walls, and 3) a single wing oscillating harmonically in freestream.

Then a common method for computing the critical flutter speed of these structures is given. The computation of critical flutter speed is based on the assumption of coupled flexural-torsional oscillation of the blades. This paper presents, therefore, a method for investigating the most general flutter phenomenon which can occur at low incidence. However, the method includes also the flutter phenomenon discovered by Whitehead[10] (purely torsional flutter) as a particular case.

The method presented here is independent of that given by Whitehead,[10] the derivation of which is also based on the solution of the Lane-Friedman integral equations. The author believes that the method gives more accurate results than that due to Smith,[7] who uses an approximate kernel function in the general integral equation. The method also is capable of giving numerical solution for the sound generation problem treated in Refs. 7 and 10.

Appendix A Single Wing Oscillating Harmonically in Freestream—Accuracy Checks

It is impossible, in this limited space, to treat this subject completely for the following reasons:

a) Since $\zeta^{1/2} < k/\beta^2$ $(0<M<1)$, the Taylor series in Eq. (17) converge for $(k/\beta^2) < N$, where N is an arbitrary positive large number. Therefore, the method holds also for large value of k.

b) N can be computed from the condition that the maximum value of independent finite integrals, whose denominators contain the quantity z^2, should not exceed a small value of $\bar{\epsilon}$. For example, the results given in Tables 1 and 2 have been computed with $\bar{\epsilon} \doteq 0.00005$.

c) Blanch and Fettis[2] have published a table for comparing the values of oscillatory force and moment coefficients tabulated according to the methods of different authors at $M = 0.7$. The aim of this section is to compare some of our results with those given in that table. From Table 1 it can be seen that, for a range of k between 0.02 and 0.7, the agreement among the results is excellent in all cases. By comparison of Eqs. (38) and (39) with these given in Ref. 2 it can be seen that

$$L_h = -C_1 \quad L_\alpha = -C_2 \quad M_h = -C_3 \quad M_\alpha = -C_4 \tag{A1}$$

where L_h to M_α are force and moment coefficients defined in Ref. 2 and C_1 to C_4 are given by Eqs. (40-43) for $a = -0.5$.

d) In Table 2, the values of some aerodynamic coefficients tabulated according to the present method are compared also with those due to Frazer.[3] From Table 2 it can be seen that for $k \leq 1$ our results are again in close agreement with those given by Frazer.

By comparing Eqs. (38) and (39) with the corresponding equations of Ref. 3, it can be seen that

$$Z_1 + iZ_2 = k^2C_1 \quad Z_3 + iZ_4 = \tfrac{1}{2}k^2C_2 \tag{A2a}$$

$$M_1 + iM_2 = \tfrac{1}{2}k^2C_3 \quad M_3 + iM_4 = \tfrac{1}{4}k^2C_4 \tag{A2b}$$

where $(Z_1 + iZ_2)$ to $(M_3 + iM_4)$ are aerodynamic coefficients defined in Ref. 3 and C_1 to C_4 are given by Eqs. (40-43) for $a = 0$.

e) From Tables 1 and 2 it is evident that for $k > 1$, our results differ from those due to Frazer, as well as due to Blanch and Fettis, but from Eqs. (13) and (17) it is evident that the present method holds also for large values of k. The corresponding results given in Tables 1 and 2 have been tabulated according to the ARTI method extended to seven collocation points $(N = 147.299)$.

Appendix B Wing Oscillating Harmonically between Two Parallel Walls

The method is checked by the results given by Woolston and Runyan.[4] The unsteady aerodynamic coefficients have been tabulated again according to the seven-point collocation method for a range of k lying between 0.022 and 0.40. The values of some coefficients are collected in Table 3.

The geometrical and physical parameters of this example are shown in Fig. 3. The dimensionless critical flutter speed vs

Table 1 Comparison with results published by Blanche and Fettis[2] ($M=0.7$ and $a=-0.5$)

	k	ARTI Re	ARTI Im	Reissner-Haskind Re	Reissner-Haskind Im	Dietze Re	Dietze Im
L_h	0.060	−9.41583	−36.7611	−9.4182	−36.762	−9.4417	−36.778
	0.100	−5.8803	−19.4439	−5.8817	−19.445	−5.8920	−19.410
	0.500	−0.21310	−2.72892	−0.2132	−2.7290	−0.21520	−2.7280
	0.700	0.03389	−1.99385	0.03390	−1.9938	0.03143	−1.9900
	2.914	0.02427	−0.55647	0.09410	−0.59994		
L_α	0.060	−623.258	120.408	−623.28	120.45	−623.31	120.08
	0.100	−201.382	39.6295	−201.39	39.642	−201.37	39.690
	0.500	−6.48164	−1.94359	−6.4819	−1.9435	−6.4824	−1.9448
	0.700	−3.55339	−1.59519	−3.5532	−1.5952	−3.5537	−1.5986
	2.914	−0.28555	−0.20103	−0.21993	−0.27851		
M_h	0.060	1.15778	−0.23861	1.1577	−0.23874	1.1556	−0.2361
	0.100	1.0629	−0.2701	1.0630	−0.27025	1.0560	−0.2520
	0.500	0.81067	−0.35915	0.81069	−0.35917	0.8110	−0.3586
	0.700	0.73875	−0.44716	0.73877	−0.44706	0.7400	−0.4447
	2.914	−0.35070	−0.34357	0.10817	−0.35372		
M_α	0.060	−3.17111	−31.2256	−3.1732	−31.226	−3.1611	−31.194
	0.100	−2.00000	−17.9348	−2.0009	−17.935	−1.9990	−17.920
	0.500	−0.40829	−3.40214	−0.40831	−3.4022	−0.3990	−3.4076
	0.700	−0.45951	−2.43890	−0.45916	−2.4388	−0.4542	−2.4463
	2.914	−0.24607	−0.27165	−0.03856	−0.26469		

Table 2 Comparison with results published by Frazer[3] ($M=0.7$ and $a=0$)

	k	ARTI Re	ARTI Im	Frazer Re	Frazer Im	
Z_1+iZ_2	0.10	0.058803	0.19444	0.05883	0.19445	F.3[a]
	0.50	0.05328	0.68223	0.05359	0.6818	F.5
	1.00	−0.08834	1.45411	−0.08949	1.450	F.7
	2.00	−0.65378	3.10111	−0.6521	3.306	F.3
Z_3+iZ_4	0.10	0.99221	−0.24676	0.9922	−0.2470	F.3
	0.50	0.79688	0.72391	0.7962	0.07183	F.5
	1.00	0.98750	0.13579	0.9837	0.1359	F.7
	2.00	1.03606	0.08211	1.137	0.1722	F.3
M_1+iM_2	0.10	−0.02002	−0.04726	−0.02002	−0.04726	F.3
	0.50	−0.11465	−0.12566	−0.1145	−0.1257	F.5
	1.00	−0.25986	−0.091597	−0.2592	−0.09270	F.7
	2.00	−0.26307	0.00353	−0.4108	−0.05466	F.3
M_3+iM_4	0.10	−0.24172	0.10619	−0.2417	0.1062	F.3
	0.50	−0.14837	0.18331	−0.1483	0.1830	F.5
	1.00	−0.04492	0.29031	−0.04599	0.2899	F.7
	2.00	0.02068	0.26515	−0.06005	0.3068	F.3

[a] Frazer's results calculated according to three-point collocation method, etc.

Table 3 Sample table of aerodynamic coefficients for a wing oscillating harmonically between two parallel walls ($M=0.7$)

k	A_{OF} Re	A_{OF} Im	A_{1F} Re	A_{1F} Im	A_{2F} Re	A_{2F} Im
0.020	2.84016	−0.28402	0.15836	0.09977	0.00476	−0.00183
0.060	2.60306	−0.75949	0.20681	0.28599	0.00972	−0.00673
0.139	1.94489	−1.19142	0.37851	0.57531	0.02934	−0.02596
0.400	−0.18094	−0.71811	1.28853	−0.59158	−0.21585	−0.25567

k	A_{0M} Re	A_{0M} Im	A_{1M} Re	A_{1M} Im	A_{2M} Re	A_{2M} Im
0.020	−0.01876	−0.13856	2.88007	−0.00782	−0.00010	−0.02841
0.060	−0.14898	−0.36561	2.90588	−0.03164	−0.00108	−0.08593
0.139	−0.51780	−0.53604	2.99517	−0.13232	−0.00934	−0.20485
0.400	−1.42299	0.44693	2.55711	−1.73981	−0.35183	−0.51431

Table 4 Appropriate values of aerodynamic coefficients for oscillating airfoils in a staggered cascade ($M=0.7$)

		A_{0F}		A_{1F}		A_{2F}	
k	$\bar{\beta}°$	Re	Im	Re	Im	Re	Im
0.300	45.0	2.00398	−0.58489	−0.37695	−0.28261	−0.50076	−0.00752
	72.0	2.41812	−0.65341	0.64946	−0.79424	−0.22185	−0.32820
	99.0	2.59870	−0.89699	2.06926	−0.94825	0.23440	−0.66866
	126.0	2.58163	−1.25954	3.60381	−0.71575	0.72446	−1.01508
	153.0	2.39609	−1.69396	4.99866	−0.15363	1.14235	−1.33912
	180.0	2.07238	−2.12676	5.95593	0.57325	1.41408	−1.56693
	207.0	1.65867	−2.45285	6.16609	1.28205	1.49878	−1.58748
	234.0	1.22915	−2.56961	5.50337	1.85206	1.39157	−1.33388
	261.0	0.85244	−2.41969	4.16353	2.18361	1.11477	−0.86499
	288.0	0.56114	−2.00143	2.56113	2.17722	0.72853	−0.35441
	315.0	0.36400	−1.36006	1.13318	1.78817	0.34288	−0.00193

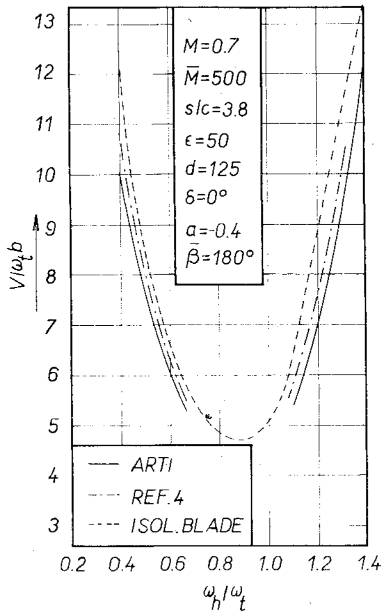

Fig. 3 Sketch illustrating the resonance phenomenon.

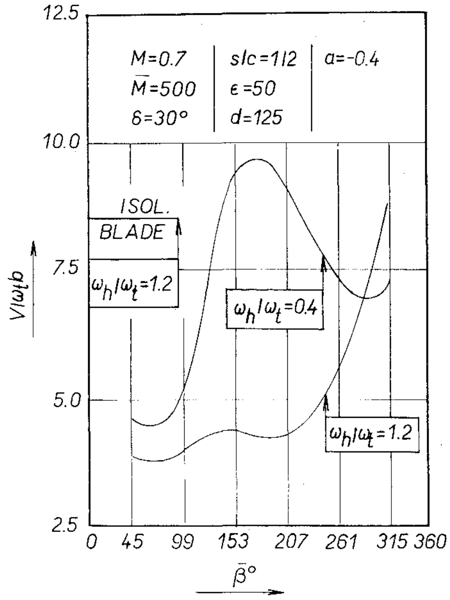

Fig. 4 Dimensionless flutter speed vs interblade phase-lag angle.

ω_h/ω_t, together with the results of Ref. 4, is plotted in Fig. 3. From Fig. 3 it can be seen that the agreement between the results calculated according to these different methods is very good.

Appendix C Oscillating Airfoil in Staggered Cascade

For this structure, the coefficients A_{0F} to A_{2M} have been tabulated for a range of k lying between 0.02 and 0.7 and for 11 values of $\bar{\beta}$ in all cases. Table 4 contains some results of computations, that have been carried out by the three-point collocation method. Figure 4 illustrates the geometrical and physical parameters of this cascade, as well as the plot of two curves ($\omega_h/\omega_t=0.4$ and 1.2) corresponding to the Lane's minimization procedure. This type of flutter must be studied at the same times as the resonance phenomenon treated in detail in Ref. 1.

It is of interest to note that the Lane's minimization procedure was first carried out by Wang, Lane, Vaccaro,[12] but their investigations hold only for a two-dimensional incompressible flow. The minimization procedure shown in Fig. 4 represents, therefore, the counterpart to Fig. 5 of Ref. 12 in two-dimensional compressible flow. It is of interest to note that Whitehead[13,14] discovered two special cases of the flutter phenomenon in cascade blades oscillating harmonically with constant phase lag angle between adjacent blades in two-dimensional incompressible flow. In fact, Ref. 13 gives conditions under which uncoupled bending flutter can occur at high angles of attack. Moreover, Ref. 14 reports the case of an uncoupled torsional flutter in incompressible flow. For the sake of completeness, it may be pointed out that Whitehead[10] discovered the existence of uncoupled torsional flutter in compressible subsonic flow.

The method given in Ref. 10 also is based on the solution of the Lane-Friedman integral equation. However, Whitehead did not add and subtract the single-wing term in the general integral equation [see Eq. (34) of Ref. 10] and, therefore, the present paper is independent of the method used in Ref. 10. The correctness of the program have been checked by comparison with some results collected in Table 1 of Ref. 7. For example for $n=5$, $s/c=1$, $\delta=30°$, $k=0.631/2$, $\bar{\beta}=\pi/2$ and $M=0.8$, Ref. 7 gives the following results $C_{Lb}=1.0264-0.7684i$, whereas our program gives $C_{Lb}=1.09769-1.0422i$. Furthermore, for $n=4$, $\delta=60°$, $S/C=1, k=0.5$, $\bar{\beta}=\pi/2$, and $M=0.1$, Ref. 7 gives $C_{Lb}=2.5683-0.1070i$, whereas the ARTI-program gives $C_{Lb}=2.50481-0.23632i$. Our results have been tabulated approximately by the three-point collocation method. Since Smith uses an approximate kernel function in the general integral equation, the author believes that the present method is somewhat more accurate, than that given in Ref. 7.

The present program has been checked also with all results given in Table 4 of Ref. 1 and excellent agreement has been obtained in all cases.

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